J. Math. Anal. Appl. 389 (2012) 781–791

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# The sharp estimates of homogeneous expansions for the generalized class of close-to-quasi-convex mappings ☆

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ARTICLE INFO

*Article history:*
Received 26 September 2011
Available online 13 December 2011
Submitted by A.V. Isaev

*Keywords:*
Homogeneous expansions
Close-to-quasi-convex mappings
Strongly close-to-quasi-convex mappings of type $\alpha$ and order $\beta$

ABSTRACT

In this paper, the class of strongly close-to-quasi-convex mappings of type $\alpha$ and order $\beta$ is introduced in the unit ball of complex Banach space or unit polydisk in $\mathbb{C}^n$, and we obtain the sharp estimates of homogeneous expansions for this class. These results generalize many known results.



## 1. Introduction and preliminaries

Let $X$ be a complex Banach space with the norm $\|.\|$, let $X^*$ be the dual space of $X$, let $E$ be the open unit ball in $X$, and $U$ represent the Euclidean open unit disk in $\mathbb{C}$. Also, let $U^n$ denote the open unit polydisk in $\mathbb{C}^n$ and let $\mathbb{N}$ be the set of all positive integers. Let the symbol $'$ mean transpose. For each $x \in X \backslash \{0\}$, we define

$$T(x) = \left\{T_x \in X^*\colon \|T_x\| = 1,\ T_x(x) = \|x\|\right\}.$$

By the Hahn–Banach theorem, $T(x)$ is nonempty.

It is well known that there is the Bieberbach conjecture (i.e., de Branges theorem) in one complex variable, but this theorem is no longer true in several complex variables [1]. Therefore, it is necessary to require some additional properties of family of mappings in order to obtain some positive results, for instance, the convexity, the starlikeness and so on.

In [2], Gong Sheng has posed the following conjecture for starlike mappings on $U^n$ in $\mathbb{C}^n$.

**Conjecture A.** *If $f: U^n \to \mathbb{C}^n$ is a normalized biholomorphic starlike mapping, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant m\|z\|^m, \quad z \in U^n,\ m = 2, 3, \ldots.$$

☆ This work was supported by NNSF of China (Grant Nos. 10971063, 11061015, 11031008), the Jiangxi Provincial Natural Science Foundation of China (Grant No. 2010GZS0096), Natural Science Foundation of Department of Education of Jiangxi Province, China (Grant No. GJJ 09149) and the Zhejiang Provincial Natural Science Foundation of China (Grant No. Y6110053).

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2011.12.023

At present, only the cases of $m = 2, 3$ (see [2,9]) have been proved. However, Roper and Suffridge [17] have proved that the above conjecture does not hold in general for normalized biholomorphic starlike mappings on the Euclidean unit ball $B^n$ in $\mathbb{C}^n$. So, in [4] (see also [6]), Graham, Hamada and Kohr proposed the following conjecture.

**Conjecture B.** *If $f : E \to \mathbb{C}^n$ is a normalized biholomorphic starlike mapping, then*

$$\frac{|T_x(D^m f(0)(x^m))|}{m!} \leqslant m\|x\|^m, \quad x \in E,\ T_x \in T(x),\ m = 2, \ldots,$$

*where $E$ is the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm.*

Conjecture B has been shown by Graham, Hamada and Kohr [4] in the case of $m = 2$.

The above conjectures can be regarded as generalizations of the Bieberbach conjecture to several complex variables, and have become one of the main researching objects in geometric function theory of several complex variables. After that, many mathematicians investigate estimates of homogeneous expansions for various interesting subclasses of holomorphic mappings. Recently, some best-possible results concerning the coefficient estimates for subclasses of holomorphic mappings in several variables were obtained in work of Graham, Hamada and Kohr [4], Graham, Kohr and Kohr [5], Hamada, Honda and Kohr [8], Hamada and Honda [9], Kohr [10], Liu and Liu [12–15], and Xu and Liu [19,20].

In this paper, stimulated by the above-cited works, we introduce a subclass of holomorphic mappings in higher dimensions which is the generalization of a subclass of close-to-convex functions in one variable, and we obtain the sharp estimates of homogeneous expansions for this class.

Let $H(E)$ denote the set of all holomorphic mappings from $E$ into $X$. It is well known that if $f \in H(E)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)\big((y-x)^n\big),$$

for all $y$ in some neighborhood of $x \in E$, where $D^n f(x)$ is the $n$th Fréchet derivative of $f$ at $x$, and for $n \geqslant 1$,

$$D^n f(x)\big((y-x)^n\big) = D^n f(x)(\underbrace{y-x, \ldots, y-x}_{n}).$$

Furthermore, $D^n f(x)$ is a bounded symmetric $n$-linear mapping from $\prod_{j=1}^{n} X$ into $X$.

A holomorphic mapping $f : E \to X$ is said to be biholomorphic if the inverse $f^{-1}$ exists and is holomorphic on the open set $f(E)$. A mapping $f \in H(E)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in E$. If $f : E \to X$ is a holomorphic mapping, then $f$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where $I$ represents the identity operator from $X$ into $X$. Let $S(E)$ be the set of all normalized biholomorphic mappings on $E$.

Suppose that $\Omega \in \mathbb{C}^n$ is a bounded circular domain. The first Fréchet derivative and the $m$ ($m \geqslant 2$)th Fréchet derivative of a mapping $f \in H(\Omega)$ at point $z \in \Omega$ are denoted by $Df(z)$, $D^m f(z)(a^{m-1}, .)$, respectively. The matrix representations are

$$Df(z) = \left(\frac{\partial f_p(z)}{\partial z_k}\right)_{1 \leqslant p,k \leqslant n}, \qquad D^m f(z)\big(a^{m-1}, .\big) = \left(\sum_{l_1, l_2, \ldots, l_{m-1}=1}^{n} \frac{\partial^m f_p(z)}{\partial z_k \partial z_{l_1} \cdots \partial z_{l_{m-1}}} a_{l_1} \cdots a_{l_{m-1}}\right)_{1 \leqslant p,k \leqslant n},$$

where $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))'$, $a = (a_1, a_2, \ldots, a_n)' \in \mathbb{C}^n$.

We first recall the following definition due to Suffridge [18].

**Definition 1.** Let $f : E \to X$ be a holomorphic mapping. We say that $f$ is close-to-convex if there exists a convex mapping $g \in H(E)$ such that

$$\Re e\big\{T_u\big[Df(x)\big(Dg(x)\big)^{-1}u\big]\big\} > 0, \quad x \in E \setminus \{0\},\ u \in X \setminus \{0\}. \tag{1}$$

In one variable, the relation (1) is equivalent to $\Re e \frac{f'(z)}{g'(z)} > 0$, $z \in U$. Therefore, Definition 1 is the usual definition of close-to-convex functions on $U$.

**Definition 2.** (See [21].) Suppose that $f : E \to X$ is a normalized locally biholomorphic mapping. If

$$\Re e\big\{T_x\big[\big(Df(x)\big)^{-1}\big(f(x) - f(\xi x)\big)\big]\big\} \geqslant 0, \quad x \in E,\ \xi \in \overline{U},$$

then $f$ is said to be a quasi-convex mapping on $E$.

Let $Q(E)$ denote the class of quasi-convex mappings on $E$.

**Definition 3.** (See [13].) Suppose that $\alpha \in [0,1)$ and $f: E \to X$ is a normalized locally biholomorphic mapping. If

$$\Re e\{T_x[(Df(x))^{-1}(D^2 f(x)(x^2) + Df(x)x)]\} \geqslant \alpha\|x\|, \quad x \in E,$$

then $f$ is said to be a quasi-convex mapping of type $B$ and order $\alpha$ on $E$.

When $\alpha = 0$, Definition 3 is the definition of the quasi-convex mapping of type $B$, which was introduced by Roper and Suffridge [17]; when $X = \mathbb{C}$, $E = U$, Definition 3 reduces to the definition of the normalized convex functions of order $\alpha$ on $U$.

Let $Q_B^\alpha(E)$ denote the class of quasi-convex mappings of type $B$ and order $\alpha$ on $E$, and let $Q_B(E)$ be the class of quasi-convex mappings of type $B$ on $E$.

Let $K_\alpha(U)$ denote the class of normalized convex functions of order $\alpha$ on $U$. Now, we define a subclass of close-to-convex functions on $U$.

**Definition 4.** Suppose that $\alpha \in [0,1)$, $\beta \in (0,1]$ and $f: U \to \mathbb{C}$ is a normalized locally biholomorphic function. If there exists a function $g \in K_\alpha(U)$ such that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leqslant \frac{\beta\pi}{2}, \quad z \in U,$$

then $f$ is said to be a strongly close-to-convex function of type $\alpha$ and order $\beta$ on $U$.

When $\alpha = 0$, Definition 4 was originally introduced by Pommerenke [16].

Combining Definition 3, we extend Definition 4 to higher dimensions as follows.

**Definition 5.** Suppose that $\alpha \in [0,1)$, $\beta \in (0,1]$ and $f: E \to X$ is a normalized holomorphic mapping. If there exists a mapping $g \in Q_B^\alpha(E)$ such that

$$\left|\arg \frac{1}{\|x\|} T_x[(Dg(x))^{-1} Df(x)x]\right| \leqslant \frac{\pi}{2}\beta, \quad x \in E, \tag{2}$$

then $f$ is said to be a strongly close-to-quasi-convex mapping of type $\alpha$ and order $\beta$ on $E$.

When $X = \mathbb{C}$, $E = U$, the relation (2) is equivalent to $|\arg \frac{f'(z)}{g'(z)}| \leqslant \frac{\beta\pi}{2}$, $z \in U$, namely this is the definition of strongly close-to-convex functions of type $\alpha$ and order $\beta$ on $U$.

**Definition 6.** (See [14].) Suppose that $f: E \to X$ is a normalized holomorphic mapping. If there exists a mapping $g \in Q(E)$ such that

$$\Re e\{T_x[(Dg(x))^{-1} Df(x)x]\} \geqslant 0, \quad x \in E, \tag{3}$$

then $f$ is said to be a close-to-quasi-convex mapping on $E$.

Let $C_\alpha^\beta(U)$ denote the class of strongly close-to-convex functions of type $\alpha$ and order $\beta$ on $U$, let $CQ_\alpha^\beta(E)$ be the class of strongly close-to-quasi-convex mappings of type $\alpha$ and order $\beta$ on $E$, the class $CQ_0^1(E)$ will be denoted by $CQ_B(E)$, and let $CQ(E)$ be the class of close-to-quasi-convex mappings on $E$.

**Remark 1.** In [3], it was proved that $Q(E) \subset Q_B(E)$, according to Definitions 5 and 6, we deduce that $CQ(E) \subset CQ_B(E)$.

**Definition 7.** (See [7].) Let $f \in H(E)$. It is said that $f$ is $k$-fold symmetric if $\exp(-(2\pi i)/k) f(e^{(2\pi i)/k}x) = f(x)$ for all $x \in E$, where $k \in \mathbb{N}$ and $i = \sqrt{-1}$.

**Definition 8.** (See [11].) Suppose that $\Omega$ is a domain (connected open set) in $X$ which contains 0. It is said that $x = 0$ is a zero of order $k$ of $f(x)$ if $f(0) = 0, \ldots, D^{k-1}f(0) = 0$, but $D^k f(0) \neq 0$, where $k \in \mathbb{N}$.

In view of Definitions 7 and 8, we see that $x = 0$ is a zero of order $k+1 (k \in \mathbb{N})$ of $f(x) - x$ if $f$ is a $k$-fold symmetric normalized holomorphic mapping $f(x) (f(x) \not\equiv x)$ defined on $E$. However, the converse is invalid.

We organize our paper as follows. In Section 2, we shall discuss the construction of strongly close-to-quasi-convex mappings of type $\alpha$ and order $\beta$. In Section 3, we shall establish sharp estimates of homogeneous expansions for the classes of $Q_B^\alpha(U^n)$ and $CQ_\alpha^\beta(U^n)$, which satisfy certain conditions. The results presented here would generalize many known results.

## 2. The construction of strongly close-to-quasi-convex mappings of type $\alpha$ and order $\beta$

The following lemmas are given to obtain the desired example in this section.

**Lemma 1.** *(See [13].) Suppose that $\alpha \in [0,1)$, and $g \in S(U)$. Then $G$ defined by $G(x) = \frac{g(T_u(x))}{T_u(x)}x$, where $\|u\| = 1$, belongs to $Q_B^{\alpha}(E)$ if and only if $g \in K_\alpha(U)$.*

**Lemma 2.** *If $f \in C_\alpha^\beta(U)$, then $F$ defined by $F(x) = \frac{f(T_u(x))}{T_u(x)}x$, where $\|u\| = 1$, belongs to $CQ_\alpha^\beta(E)$.*

**Proof.** Since $f \in C_\alpha^\beta(U)$, there exists a function $g \in K_\alpha(U)$ such that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leqslant \frac{\beta\pi}{2}, \quad z \in U. \tag{4}$$

By Lemma 1, we have $G(x) = \frac{g(T_u(x))}{T_u(x)}x \in Q_B^{\alpha}(E)$. Straightforward calculation yields

$$\big(DG(x)\big)^{-1}\eta = \frac{1}{h(x)}\left[\eta - \frac{(Dh(x)\eta)x}{h(x) + Dh(x)x}\right], \quad \eta \in X,$$

where $h(x) = \frac{g(T_u(x))}{T_u(x)}$.

Also

$$DF(x)x = f'\big(T_u(x)\big)x, \qquad Dh(x)DF(x)x = f'\big(T_u(x)\big)Dh(x)x,$$

and

$$h(x) + Dh(x)x = g'\big(T_u(x)\big).$$

Hence

$$\big(DG(x)\big)^{-1}DF(x)x = \frac{1}{h(x)}\left(DF(x)x - \frac{(Dh(x)DF(x)x)x}{h(x) + Dh(x)x}\right) = \frac{f'(T_u(x))x}{g'(T_u(x))}. \tag{5}$$

By (4) and (5), we deduce that

$$\left|\arg \frac{1}{\|x\|}T_x\big[\big(DG(x)\big)^{-1}DF(x)x\big]\right| = \left|\arg \frac{f'(T_u(x))}{g'(T_u(x))}\right| \leqslant \frac{\pi}{2}\beta, \quad x \in E. \tag{6}$$

Hence, by (6) and Definition 5, we obtain that $F \in CQ_\alpha^\beta(E)$. $\square$

**Example 1.** Suppose $\alpha \in [0,1)$, $\beta \in (0,1]$. If

$$f(z) = \left(\int_0^{z_1} \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+(1-2\beta)t}{1-t}\right)dt, \frac{z_2}{z_1}\int_0^{z_1} \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+(1-2\beta)t}{1-t}\right)dt, \ldots, \right.$$
$$\left. \frac{z_n}{z_1}\int_0^{z_1} \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+(1-2\beta)t}{1-t}\right)dt\right)',$$

then $f \in CQ_\alpha^\beta(U^n)$.

**Proof.** First, we prove that

$$f(z) = \int_0^z \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+t}{1-t}\right)^\beta dt \in C_\alpha^\beta(U), \quad z \in U.$$

For this purpose, we take

$$g(z) = \int_0^z \frac{1}{(1-t)^{2(1-\alpha)}}\,dt, \quad z \in U,$$

it is obvious that $g \in K_\alpha(U)$. A simple computation shows that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| = \left|\arg\left(\frac{1+t}{1-t}\right)^\beta\right| \leqslant \frac{\beta\pi}{2}, \quad z \in U.$$

From Definition 4, $f \in C_\alpha^\beta(U)$.

According to Lemma 2, we obtain

$$z\int_0^{T_u(z)} \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+t}{1-t}\right)^\beta dt/T_u(z) \in CQ_\alpha^\beta\left(U^n\right),$$

where $\|u\| = 1$.

By taking $u = (1, 0, \dots, 0)'$, notice that $T_u = (1, 0, \dots, 0)$, we have $T_u(z) = z_1$. Hence, we have

$$f(z) = \left(\int_0^{z_1} \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+t}{1-t}\right)^\beta dt, \frac{z_2}{z_1}\int_0^{z_1} \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+t}{1-t}\right)^\beta dt, \dots,\right.$$
$$\left.\frac{z_n}{z_1}\int_0^{z_1} \frac{1}{(1-t)^{2(1-\alpha)}}\left(\frac{1+t}{1-t}\right)^\beta dt\right)' \in CQ_\alpha^\beta\left(U^n\right),$$

and the proof is complete. □

## 3. The sharp estimates of homogeneous expansions for strongly close-to-quasi-convex mappings of type $\alpha$ and order $\beta$

In order to obtain desired theorems in this section, we need to establish the following lemmas. It is not difficult to prove Lemmas 3 and 4 (the details of the proof are omitted here).

**Lemma 3.** *Suppose that $\alpha \in [0, 1)$, and $g$ is a normalized locally biholomorphic mapping on $U^n$. Then $g \in Q_B^\alpha(U^n)$ if and only if*

$$\Re e \frac{h_j(z)}{z_j} \geqslant \alpha, \quad z = (z_1, \dots, z_n)' \in U^n,$$

*where $h(z) = (h_1(z), \dots, h_n(z))' = (Dg(z))^{-1}(D^2 g(z)(z^2) + Dg(z)z)$, $z \in U^n$, is a column vector in $\mathbb{C}^n$ and $j$ satisfies $|z_j| = \|z\| = \max_{1 \leqslant k \leqslant n}\{|z_k|\}$.*

**Lemma 4.** *Suppose that $\alpha \in [0, 1)$, $\beta \in (0, 1]$, and $f$ is a normalized holomorphic mapping on $U^n$. Then $f \in CQ_\alpha^\beta(U^n)$ if and only if*

$$\left|\arg \frac{p_j(z)}{z_j}\right| \leqslant \frac{\pi}{2}\beta, \quad z = (z_1, \dots, z_n)' \in U^n,$$

*where $p(z) = (p_1(z), \dots, p_n(z))' = (Dg(z))^{-1} Df(z)z$, $z \in U^n$, is a column vector in $\mathbb{C}^n$, $j$ satisfies $|z_j| = \|z\| = \max_{1 \leqslant k \leqslant n}\{|z_k|\}$ and $g \in Q_B^\alpha(U^n)$.*

**Lemma 5.** *(See [19].) Suppose that $\alpha \in [0, 1)$, and $h(z) = (h_1(z), \dots, h_n(z))'$ is a normalized holomorphic mapping on $U^n$. If $\Re e[\frac{h_j(z)}{z_j}] \geqslant \alpha$, $z \in U^n$, where $|z_j| = \|z\| = \max_{1 \leqslant k \leqslant n}\{|z_k|\}$, then*

$$\frac{\|D^m h(0)(z^m)\|}{m!} \leqslant 2(1-\alpha)\|z\|^m, \quad m = 2, 3, \dots.$$

**Lemma 6.** *(See [19].) Suppose that $\beta \in (0, 1]$, and $p(z) = (p_1(z), \dots, p_n(z))'$ is a normalized holomorphic mapping on $U^n$. If $|\arg \frac{p_j(z)}{z_j}| \leqslant \frac{\pi}{2}\beta$, $z \in U^n$, where $|z_j| = \|z\| = \max_{1 \leqslant k \leqslant n}\{|z_k|\}$, then*

$$\frac{\|D^m p(0)(z^m)\|}{m!} \leqslant 2\beta\|z\|^m, \quad m = 2, 3, \dots.$$

**Lemma 7.** *(See [15].) Suppose that $g \in H(U^n)$, and*

$$\frac{D^m g_p(0)(z^m)}{m!} = \sum_{l_1, l_2, \dots, l_m = 1}^{n} a_{p l_1 l_2 \cdots l_m} z_{l_1} z_{l_2} \cdots z_{l_m}, \quad p = 1, 2, \dots, n,$$

where $a_{pl_1l_2\cdots l_m} = \frac{1}{m!}\frac{\partial^m g_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_m}}$, $l_1, l_2, \ldots, l_m = 1, 2, \ldots, n$, $m = 2, 3, \ldots$. Then

$$\frac{1}{m!}D^m g_p(0)\big(z^{m-1}, w\big) = \frac{1}{m}\left(\sum_{l_1,l_2,\ldots,l_m=1}^{n} a_{pl_1l_2\cdots l_m} w_{l_1} z_{l_2}\cdots z_{l_m} + \sum_{l_1,l_2,\ldots,l_m=1}^{n} a_{pl_1l_2\cdots l_m} z_{l_1} w_{l_2} z_{l_3}\cdots z_{l_m} + \cdots + \sum_{l_1,l_2,\ldots,l_m=1}^{n} a_{pl_1l_2\cdots l_m} z_{l_1} z_{l_2}\cdots z_{l_{m-1}} w_{l_m}\right), \quad z \in U^n,\ p = 1, 2, \ldots, n,\ m = 2, 3, \ldots,$$

where $w = (w_1, w_2, \ldots, w_n)' \in \mathbb{C}^n$ satisfies $\|w\| = \max_{1\leqslant p\leqslant n}\{|w_p|\} < 1$.

**Lemma 8.** *(See [15].) Suppose $g$ is a normalized locally biholomorphic mapping on $U^n$. If $h(z) = (Dg(z))^{-1}(D^2g(z)(z^2) + Dg(z)z)$, then*

$$\frac{D^2g(0)(z^2)}{2!} = \frac{1}{2}\frac{D^2h(0)(z^2)}{2!},$$

*and*

$$m(m-1)\frac{D^m g(0)(z^m)}{m!} = \frac{D^m h(0)(z^m)}{m!} + 2\frac{D^2g(0)(z, \frac{D^{m-1}h(0)(z^{m-1})}{(m-1)!})}{2!} + \cdots + \frac{(m-1)D^{m-1}g(0)(z^{m-2}, \frac{D^2h(0)(z^2)}{2!})}{(m-1)!}, \quad z \in U^n,\ m = 3, 4, \ldots.$$

**Lemma 9.** *(See [14].) Suppose $f$ is a normalized holomorphic mapping on $U^n$ and $g$ is a normalized locally biholomorphic mapping on $U^n$. If $p(z) = (Dg(z))^{-1}Df(z)z \in H(U^n)$, then*

$$2\frac{D^2f(0)(z^2)}{2!} = \frac{D^2p(0)(z^2)}{2!} + 2\frac{D^2g(0)(z^2)}{2!} \tag{7}$$

*and*

$$m\frac{D^m f(0)(z^m)}{m!} = \frac{D^m p(0)(z^m)}{m!} + 2\frac{D^2g(0)(z, \frac{D^{m-1}p(0)(z^{m-1})}{(m-1)!})}{2!} + \cdots \tag{8}$$

$$+ (m-1)\frac{D^{m-1}g(0)(z^{m-2}, \frac{D^2p(0)(z^2)}{2!})}{(m-1)!} + m\frac{D^m g(0)(z^m)}{m!}, \quad m = 3, 4, \ldots. \tag{9}$$

**Lemma 10.** *(See [15].) Let*

$$\left\|\begin{pmatrix} \sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{1l_1l_2\cdots l_m}| e^{i\frac{\theta_{1l_1}+\theta_{1l_2}+\cdots+\theta_{1l_m}}{m}} z_{l_1}z_{l_2}\cdots z_{l_m} \\ \sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{2l_1l_2\cdots l_m}| e^{i\frac{\theta_{2l_1}+\theta_{2l_2}+\cdots+\theta_{2l_m}}{m}} z_{l_1}z_{l_2}\cdots z_{l_m} \\ \vdots \\ \sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{nl_1l_2\cdots l_m}| e^{i\frac{\theta_{nl_1}+\theta_{nl_2}+\cdots+\theta_{nl_m}}{m}} z_{l_1}z_{l_2}\cdots z_{l_m} \end{pmatrix}\right\| \leqslant C_m\|z\|^m, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in U^n,$$

*where $m = 2, 3, \ldots$, each $a_{pl_1l_2\cdots l_m}$ $(p, l_1, l_2, \ldots, l_m = 1, 2, \ldots, n)$ is a complex number which is independent of $z_p$ $(p = 1, 2, \ldots, n)$, $i = \sqrt{-1}$, each $\theta_{pl_q} \in (-\pi, \pi]$ $(q = 1, 2, \ldots, m;\ p, l_1, l_2, \ldots, l_m = 1, 2, \ldots, n)$ which is independent of $z_p$ $(p = 1, 2, \ldots, n)$, $\|z\| = \max_{1\leqslant p\leqslant n}\{|z_p|\}$, and each $C_m$ $(m = 2, 3, \ldots)$ is a nonnegative real constant which is only dependent on $m$. Then*

$$A_m = \max_{1\leqslant p\leqslant n}\left\{\sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{pl_1l_2\cdots l_m}|\right\} \leqslant C_m, \quad m = 2, 3, \ldots.$$

Using the method similar to that of [15], Theorem 1 can be proved (the details of the proof are omitted here).

**Theorem 1.** *If $g \in Q_B^\alpha(U^n)$, and*

$$\frac{D^s g_p(0)(z^s)}{s!} = \sum_{l_1,l_2,\ldots,l_m=1}^{n} |a_{pl_1l_2\cdots l_s}| e^{i\frac{\theta_{pl_1}+\theta_{pl_2}+\cdots+\theta_{pl_s}}{s}} z_{l_1}z_{l_2}\cdots z_{l_s}, \quad p = 1, 2, \ldots, n,$$

*where* $|a_{pl_1l_2\cdots l_s}|e^{i\frac{\theta_{pl_1}+\theta_{pl_2}+\cdots+\theta_{pl_s}}{s}} = \frac{1}{s!}\frac{\partial^s g_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_s}}$, $i=\sqrt{-1}$, $\theta_{pl_q}\in(-\pi,\pi]$ $(q=1,2,\ldots,s)$, $l_1,l_2,\ldots,l_s=1,2,\ldots,n$, $s=2,3,\ldots,m-1$, *then*

$$\frac{\|D^m g(0)(z^m)\|}{m!} \leqslant \frac{2(1-\alpha)}{m(m-1)}\left(1+\sum_{s=2}^{m-1} sA_s\right)\|z\|^m, \quad z\in U^n,\ m=3,4,\ldots,$$

*where* $A_s=\max_{1\leqslant p\leqslant n}\{\sum_{l_1,l_2,\ldots,l_m=1}^n |a_{pl_1l_2\cdots l_s}|\}$, $s=2,3,\ldots,m-1$.

**Remark 2.** Theorem 1 generalizes the corresponding result of [15], when $\alpha=0$, this result was obtained by Liu and Liu [15].

**Corollary 1.** *Suppose* $k\in\mathbb{N}$. *If* $g\in Q_B^\alpha(U^n)$, *and* $z=0$ *is a zero of order* $k+1$ *of* $g(z)-z$, *then*

$$\frac{\|D^{k+1}g(0)(z^{k+1})\|}{(k+1)!} \leqslant \frac{2(1-\alpha)}{(k+1)k}\|z\|^{k+1}, \quad z\in U^n.$$

*The above estimate is sharp.*

**Proof.** When $k=1$, in view of the hypothesis of Corollary 1, Lemmas 8 and 5 (the case of $m=2$), the result follows. When $k\geqslant 2$, $m=k+1$, according to the hypothesis of Corollary 1, we obtain that $A_s=0$, $s=2,3,\ldots,k$. From Theorem 1, we deduce that

$$\frac{\|D^{k+1}g(0)(z^{k+1})\|}{(k+1)!} \leqslant \frac{2(1-\alpha)}{(k+1)k}\|z\|^{k+1}, \quad z\in U^n.$$

This completes the proof. $\square$

It is not difficult to verify that

$$g(z)=\left(\int_0^{z_1}\frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\,dt,\ \frac{z_2}{z_1}\int_0^{z_1}\frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\,dt,\ \ldots,\ \frac{z_n}{z_1}\int_0^{z_1}\frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\,dt\right)', \quad z\in U^n,$$

satisfies the hypothesis of Corollary 1. Taking $z=(r,0,\ldots,0)'$ $(0\leqslant r<1)$, we have

$$\frac{\|D^{k+1}g(0)(z^{k+1})\|}{(k+1)!} = \frac{2(1-\alpha)}{(k+1)k}r^{k+1}.$$

Hence, the estimate of Corollary 1 is sharp.

**Corollary 2.** *Suppose* $k\in\mathbb{N}$. *If* $g$ *is a* $k$*-fold symmetric quasi-convex mapping of type* $B$ *and order* $\alpha$ *on* $U^n$, *and*

$$\frac{D^{tk+1}g_p(0)(z^{tk+1})}{(tk+1)!} = \sum_{l_1,l_2,\ldots,l_{tk+1}=1}^{n} |a_{pl_1l_2\cdots l_{tk+1}}|e^{i\frac{\theta_{pl_1}+\theta_{pl_2}+\cdots+\theta_{pl_{tk+1}}}{tk+1}} z_{l_1}z_{l_2}\cdots z_{l_{tk+1}}, \quad p=1,2,\ldots,n,$$

*where* $|a_{pl_1l_2\cdots l_{tk+1}}|e^{i\frac{\theta_{pl_1}+\theta_{pl_2}+\cdots+\theta_{pl_{tk+1}}}{tk+1}} = \frac{1}{(tk+1)!}\frac{\partial^{kt+1}g_p(0)}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_{tk+1}}}$, $i=\sqrt{-1}$, $\theta_{pl_q}\in(-\pi,\pi]$ $(q=1,2,\ldots,tk+1)$, $l_1,l_2,\ldots,l_{kt+1}=1,2,\ldots,n$, $t=1,2,\ldots$, *then*

$$\frac{\|D^{tk+1}g(0)(z^{tk+1})\|}{(tk+1)!} \leqslant \frac{\prod_{r=1}^t((r-1)k+2(1-\alpha))}{(tk+1)\cdot t!k^t}\|z\|^{tk+1}, \quad z\in U^n,\ t=1,2,\ldots. \tag{10}$$

*The above estimates are sharp.*

**Proof.** It is clear that if $g$ is a $k$-fold symmetric quasi-convex mapping of type $B$ and order $\alpha$ on $U^n$, then $z=0$ is a zero of order $k+1$ $(k\in\mathbb{N})$ of $g(z)-z$. In view of the hypotheses of Corollary 2 and Corollary 1, we conclude that

$$\frac{\|D^{k+1}g(0)(z^{k+1})\|}{(k+1)!} \leqslant \frac{2(1-\alpha)}{(k+1)k}\|z\|^{k+1}, \quad z\in U^n.$$

That is, (10) holds for $t=1$. Assume now that (10) holds for $t=1,2,\ldots,j$ for some integer $j\geqslant 2$. This implies

$$\frac{\|D^{tk+1}g(0)(z^{tk+1})\|}{(tk+1)!} \leqslant \frac{\prod_{r=1}^t((r-1)k+2(1-\alpha))}{(tk+1)\cdot t!k^t}\|z\|^{tk+1}, \quad z\in U^n,\ t=1,2,\ldots,j. \tag{11}$$

Taking into account (11), we set

$$C_{tk+1}=\frac{\prod_{r=1}^{t}((r-1)k+2(1-\alpha))}{(tk+1)\cdot t!k^t},\quad t=1,2,\dots,j.$$

From the hypotheses of Corollary 2, we have $A_m=0$, $2\leqslant m\neq tk+1$ $(t=1,2,\dots)$. In view of Lemma 10 and Theorem 1, we deduce that

$$\begin{aligned}\frac{\|D^{(j+1)k+1}g(0)(z^{(j+1)k+1})\|}{[(j+1)k+1]!}&\leqslant\frac{2(1-\alpha)}{[(j+1)k+1](j+1)k}\left[1+\sum_{t=1}^{j}(tk+1)A_{tk+1}\right]\|z\|^{(j+1)k+1}\\&\leqslant\frac{2(1-\alpha)}{[(j+1)k+1](j+1)k}\left[1+\sum_{t=1}^{j}(tk+1)\frac{\prod_{r=1}^{t}((r-1)k+2(1-\alpha))}{(tk+1)\cdot t!k^t}\right]\|z\|^{(j+1)k+1}\\&=\frac{2(1-\alpha)}{[(j+1)k+1](j+1)k}\left[1+\sum_{t=1}^{j}\frac{\prod_{r=1}^{t}((r-1)k+2(1-\alpha))}{t!k^t}\right]\|z\|^{(j+1)k+1}\\&=\frac{2(1-\alpha)}{[(j+1)k+1](j+1)k}\cdot\frac{\prod_{r=2}^{j+1}((r-1)k+2(1-\alpha))}{j!k^j}\|z\|^{(j+1)k+1}\\&=\frac{\prod_{r=1}^{j+1}((r-1)k+2(1-\alpha))}{((j+1)k+1)\cdot(j+1)!k^{j+1}}\|z\|^{(j+1)k+1}.\end{aligned}$$

That is, (10) holds for $t=j+1$. This completes the proof. □

It is easy to verify that

$$g(z)=\left(\int_0^{z_1}\frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\,dt,\frac{z_2}{z_1}\int_0^{z_1}\frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\,dt,\dots,\frac{z_n}{z_1}\int_0^{z_1}\frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\,dt\right)',\quad z\in U^n,$$

satisfies the hypotheses of Corollary 2. Taking $z=(r,0,\dots,0)'$ $(0\leqslant r<1)$, we have

$$\frac{\|D^{tk+1}g(0)(z^{tk+1})\|}{(tk+1)!}=\frac{\prod_{u=1}^{t}((u-1)k+2(1-\alpha))}{(tk+1)\cdot t!k^t}r^{tk+1},\quad t=1,2,\dots.$$

Hence, the estimate of Corollary 2 is sharp.

**Remark 3.** Corollary 2 generalizes the corresponding result of [13], when $l_1=p_1$, $l_2=\dots=l_{tk+1}=l$ $(l=1,2,\dots,n)$, Corollary 2 is the result of [13]. However, the methods of their proofs are different.

**Theorem 2.** *Suppose $f$ is a normalized holomorphic mapping on $U^n$ and $g$ satisfies the assumptions of Theorem 1. If $f$ is a strongly close-to-quasi-convex mapping of type $\alpha$ and order $\beta$ on $U^n$ with respect to $g$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!}\leqslant\left[\frac{2\beta}{m}+\frac{2(1-\alpha)}{m(m-1)}\right]\left(1+\sum_{s=2}^{m-1}sA_s\right)\|z\|^m,\quad z\in U^n,\ m=3,4,\dots,$$

*where $A_s$ is defined as in Theorem 1.*

**Proof.** Fix $z\in U^n\setminus\{0\}$, denote $z_0=\frac{z}{\|z\|}$. Let

$$p(z)=\big(p_1(z),\dots,p_n(z)\big)'=\big(Dg(z)\big)^{-1}Df(z)z,\quad z\in U^n.$$

Since $f\in CQ_\alpha^\beta(U^n)$, in view of Lemma 4, we obtain that

$$\left|\arg\frac{p_j(z)}{z_j}\right|\leqslant\frac{\pi}{2}\beta,\quad z=(z_1,\dots,z_n)'\in U^n,$$

where $j$ satisfies $|z_j|=\|z\|=\max_{1\leqslant k\leqslant n}\{|z_k|\}$. According to Lemma 6, we have

$$\frac{\|D^m p(0)(z^m)\|}{m!}\leqslant 2\beta\|z\|^m,\quad z\in U^n,\ m=2,3,\dots.\tag{12}$$

Denote $w = \frac{D^{m-s+1} g(0)(z^{m-s+1})}{(m-s+1)!}$, $s = 2, 3, \ldots, m-1$, and $j$ satisfies $|z_j| = \|z\| = \max_{1 \leqslant k \leqslant n}\{|z_k|\}$. By the hypotheses of Theorem 2, Lemma 7 and (12), we obtain

$$\begin{aligned}
&\left|\frac{1}{s!} D^s g_j(0)\left(z_0^{s-1}, \frac{D^{m-s+1} p(0)(z_0^{m-s+1})}{(m-s+1)!}\right)\right| \\
&= \frac{1}{s}\left|\sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{D^{m-s+1} p_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!} \frac{z_{l_2}}{\|z\|} \cdots \frac{z_{l_s}}{\|z\|}\right. \\
&\quad + \sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|} \frac{D^{m-s+1} p_{l_2}(0)(z_0^{m-s+1})}{(m-s+1)!} \frac{z_{l_3}}{\|z\|} \cdots \frac{z_{l_s}}{\|z\|} + \cdots \\
&\quad \left. + \sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|} \cdots \frac{z_{l_{s-1}}}{\|z\|} \frac{D^{m-s+1} p_{l_s}(0)(z_0^{m-s+1})}{(m-s+1)!}\right| \\
&\leqslant \frac{1}{s}\left(\left|\sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{D^{m-s+1} p_{l_1}(0)(z_0^{m-s+1})}{(m-s+1)!} \frac{z_{l_2}}{\|z\|} \cdots \frac{z_{l_s}}{\|z\|}\right|\right. \\
&\quad + \left|\sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|} \frac{D^{m-s+1} p_{l_2}(0)(z_0^{m-s+1})}{(m-s+1)!} \frac{z_{l_3}}{\|z\|} \cdots \frac{z_{l_s}}{\|z\|}\right| + \cdots \\
&\quad \left. + \left|\sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| e^{i\frac{\theta_{jl_1}+\theta_{jl_2}+\cdots+\theta_{jl_s}}{s}} \cdot \frac{z_{l_1}}{\|z\|} \cdots \frac{z_{l_{s-1}}}{\|z\|} \frac{D^{m-s+1} p_{l_s}(0)(z_0^{m-s+1})}{(m-s+1)!}\right|\right) \\
&\leqslant \frac{1}{s}\left(\sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| \cdot \frac{|D^{m-s+1} p_{l_1}(0)(z_0^{m-s+1})|}{(m-s+1)!} \frac{|z_{l_2}|}{\|z\|} \cdots \frac{|z_{l_s}|}{\|z\|}\right. \\
&\quad + \sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| \cdot \frac{|z_{l_1}|}{\|z\|} \frac{|D^{m-s+1} p_{l_2}(0)(z_0^{m-s+1})|}{(m-s+1)!} \frac{|z_{l_3}|}{\|z\|} \cdots \frac{|z_{l_s}|}{\|z\|} + \cdots \\
&\quad \left. + \sum_{l_1, l_2, \ldots, l_s=1}^{n} |a_{jl_1, l_2, \ldots, l_s}| \cdot \frac{|z_{l_1}|}{\|z\|} \cdots \frac{|z_{l_{s-1}}|}{\|z\|} \frac{|D^{m-s+1} p_{l_s}(0)(z_0^{m-s+1})|}{(m-s+1)!}\right) \\
&\leqslant \frac{1}{s}(\underbrace{2\beta A_s + 2\beta A_s + \cdots + 2\beta A_s}_{s}) = 2\beta A_s.
\end{aligned}$$

Using the similar method to that of [15], we prove that

$$\left\|\frac{1}{s!} D^s g(0)\left(z^{s-1}, \frac{D^{m-s+1} p(0)(z^{m-s+1})}{(m-s+1)!}\right)\right\| \leqslant 2\beta A_s \|z\|^m, \quad z \in U^n,\ s = 2, 3, \ldots, m-1. \tag{13}$$

When $m = 3, 4, \ldots$, according to Lemma 9, (12) and (13), we have

$$\begin{aligned}
m\frac{\|D^m f(0)(z^m)\|}{m!} &= \left\|\frac{D^m p(0)(z^m)}{m!} + 2\frac{D^2 g(0)(z, \frac{D^{m-1} p(0)(z^{m-1})}{(m-1)!})}{2!} + \cdots\right. \\
&\quad \left. + (m-1)\frac{D^{m-1} g(0)(z^{m-2}, \frac{D^2 p(0)(z^2)}{2!})}{(m-1)!} + m\frac{D^m g(0)(z^m)}{m!}\right\| \\
&\leqslant \left\|\frac{D^m p(0)(z^m)}{m!}\right\| + 2\left\|\frac{D^2 g(0)(z, \frac{D^{m-1} p(0)(z^{m-1})}{(m-1)!})}{2!}\right\| + \cdots \\
&\quad + (m-1)\left\|\frac{D^{m-1} g(0)(z^{m-2}, \frac{D^2 p(0)(z^2)}{2!})}{(m-1)!}\right\| + m\left\|\frac{D^m g(0)(z^m)}{m!}\right\| \\
&\leqslant 2\beta\|z\|^m + 2 \cdot 2\beta A_s \|z\|^m + \cdots + (m-1) \cdot 2\beta A_s \|z\|^m + \frac{2(1-\alpha)}{(m-1)}\left(1 + \sum_{s=2}^{m-1} s A_s\right)\|z\|^m.
\end{aligned}$$

That is,

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant \left[\frac{2\beta}{m} + \frac{2(1-\alpha)}{m(m-1)}\right]\left(1+\sum_{s=2}^{m-1} sA_s\right)\|z\|^m, \quad z \in U^n,\ m = 3, 4, \ldots.$$

This completes the proof. □

**Corollary 3.** *Suppose $f$ is a normalized holomorphic mapping on $U^n$. If $f$ is a strongly close-to-quasi-convex mapping of type $\alpha$ and order $\beta$ with respect to $g$, and $z = 0$ is a zero of order $k+1$ of $g(z) - z$, then*

$$\frac{\|D^{k+1} f(0)(z^{k+1})\|}{(k+1)!} \leqslant \frac{2\beta}{k+1}\|z\|^{k+1} + \frac{2(1-\alpha)}{k(k+1)}\|z\|^{k+1}, \quad z \in U^n.$$

*The above estimate is sharp.*

**Proof.** When $k = 1$, in view of the hypotheses of Corollary 3, (7) of Lemma 9, Lemmas 5 and 6, the result follows. When $k \geqslant 2$, $m = k+1$, according to the hypotheses of Corollary 3, it is known that $A_s = 0$, $s = 2, 3, \ldots, k$. From Theorem 2, we deduce

$$\frac{\|D^{k+1} f(0)(z^{k+1})\|}{(k+1)!} \leqslant \frac{2\beta}{k+1}\|z\|^{k+1} + \frac{2(1-\alpha)}{k(k+1)}\|z\|^{k+1}, \quad z \in U^n. \qquad \square$$

In order to see that the estimation of Corollary 3 is sharp, it suffices to consider the following example.

**Example 2.** Suppose

$$f(z) = \left(\int_0^{z_1} \frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\left(\frac{1+t^k}{1-t^k}\right)^{\beta} dt, \frac{z_2}{z_1}\int_0^{z_1} \frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\left(\frac{1+t^k}{1-t^k}\right)^{\beta} dt, \ldots, \frac{z_n}{z_1}\int_0^{z_1} \frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\left(\frac{1+t^k}{1-t^k}\right)^{\beta} dt\right)', \quad z \in U^n,$$

and

$$g(z) = \left(\int_0^{z_1} \frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\, dt, \frac{z_2}{z_1}\int_0^{z_1} \frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\, dt, \ldots, \frac{z_n}{z_1}\int_0^{z_1} \frac{1}{(1-t^k)^{(2(1-\alpha))/k}}\, dt\right)', \quad z \in U^n,$$

then it is not difficult to verify that $f$ is a strongly close-to-quasi-convex mapping of type $\alpha$ and order $\beta$ on $U^n$ with respect to $g$, and $z = 0$ is a zero of order $k+1$ of $g(z) - z$. Taking $z = (r, 0, \ldots, 0)'$ $(0 \leqslant r < 1)$, we have

$$\frac{\|D^{k+1} f(0)(z^{k+1})\|}{(k+1)!} = \frac{2\beta}{k+1} r^{k+1} + \frac{2(1-\alpha)}{k(k+1)} r^{k+1}.$$

Hence, the estimate of Corollary 1 is sharp.

**Corollary 4.** *Suppose that $f$ is a normalized holomorphic mapping on $U^n$ and $g$ satisfies the assumptions of Corollary 2. If $f$ is a strongly close-to-quasi-convex mapping of type $\alpha$ and order $\beta$ on $U^n$ with respect to $g$, then*

$$\frac{\|D^{kt+1} f(0)(z^{kt+1})\|}{(kt+1)!} \leqslant \left[\frac{tk\beta}{1-\alpha} + 1\right]\frac{\prod_{r=1}^{t}((r-1)k + 2(1-\alpha))}{(tk+1)\cdot t!k^t}\|z\|^{tk+1}, \quad z \in U^n,\ t = 1, 2, \ldots.$$

*The above estimates are sharp.*

**Proof.** According to Theorem 2 and using similar method to that of Corollary 2, we obtain the desired result. The example which shows that the estimations of Corollary 4 are sharp is the same as Example 2. □

Taking $k = 1$ in Corollary 4, we are led easily to Corollary 5.

**Corollary 5.** *Suppose that $f$ is a normalized holomorphic mapping on $U^n$ and $g$ satisfies the assumptions of Corollary* 2 (*the case of* $k=1$). *If $f$ is a strongly close-to-quasi-convex mapping of type $\alpha$ and order $\beta$ with respect to $g$, then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant \left[\frac{(m-1)\beta}{1-\alpha}+1\right]\frac{\prod_{r=2}^{m}(r-2\alpha)}{m!}\|z\|^m, \quad z \in U^n,\ m=2,3,\ldots. \tag{14}$$

*The above estimates are sharp.*

**Proof.** Setting $k=1$ in Corollary 4 and denoting $m=t+1$, we can deduce (14). Setting $k=1$ in Example 2, we obtain the example which shows that the estimations of Corollary 5 are sharp. □

Setting $\alpha=0$, $\beta=1$ in Corollary 5, we can readily deduce the following corollary (the proof are omitted here).

**Corollary 6.** *Suppose that $g$ satisfies the assumptions of Corollary* 2 (*the case of* $k=1$). *If $f \in CQ_B(U^n)$* (*with respect to* $g$), *then*

$$\frac{\|D^m f(0)(z^m)\|}{m!} \leqslant m\|z\|^m, \quad z \in U^n,\ m=2,3,\ldots. \tag{15}$$

**Remark 4.** In view of Remark 1, $CQ(U^n) \subset CQ_B(U^n)$, therefore, Corollary 6 generalizes the corresponding result of [14], when $g \in Q(U^n)$, and $l_1=p$, $l_2=\cdots=l_m=l$ $(l=1,2,\ldots,n)$, this result was obtained by Liu and Liu [14].

## Acknowledgment

The authors thank the referee for helpful comments and suggestions.

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